Identification of seasonality and trends for GDP with state space reduction method

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January 19, 2025

1 State space representation

In general case an autoregressive system[1] can be represented as in a state space under the form

$$\begin{aligned} x_{t+1} &= \mathbf{A} \cdot x_t + \mathbf{K} \cdot \varepsilon_t, \\ y_t &= \mathbf{C} \cdot x_t + \varepsilon_t, \end{aligned}$$

where x_t is an unobservalbe vector of $k \times 1$ state variables,
y_t is a vector $l \times 1$ observable variables,
ε_t is an ergotic strictly stationary proccess:

$$\mathsf{E}\{\varepsilon_t | \mathscr{F}_{t-1}\} = 0, \quad \lim_{k \to \infty} \mathsf{E}\{\varepsilon_t \varepsilon_t' | \mathscr{F}_{t-k}\} = \Omega_0 = \mathsf{E}\varepsilon_t \varepsilon_t' > 0,$$

$$\mathsf{E}\varepsilon_{t,j}^4 < \infty, \quad j = 1, \ldots, s,$$

with $\mathscr{F}_t = \sigma\{\varepsilon_t, \varepsilon_{t-1}, \ldots\}$ and $\varepsilon_{t,j}$ is an j entry of the vector вектора ε_t, Bauer (2008).

Eliminating ε_t we have

$$\begin{aligned} x_{t+1} &= \mathbf{A} \cdot x_t, \\ y_t &= \mathbf{C} \cdot x_t, \end{aligned}$$

where x_t is an unobservable vector of $k \times 1$ state variables,
y_t is a vector $l \times 1$ observable variables.

*This report is the personal position of the author. The results should not be considered, is the official position of the Bank of Russia.

[1]which is equivalent to ARMA representation, see 4.2 в Aoki (1987).

The Hankel matrix has the form

$$\mathbf{H} = \begin{pmatrix} y_1 & y_2 & \cdots & y_m \\ y_2 & y_3 & \cdots & y_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_{n+1} & \cdots & y_{n+m-1} \end{pmatrix} = \mathbf{\Gamma}_{1:n} \cdot \mathbf{\Omega}_{1:m}$$

and can be expressed via the matrix of observability $\mathbf{\Gamma}_{1:n}$ and the controllability matrix $\mathbf{\Omega}_{1:m}$:

$$\mathbf{\Gamma}_{1:n} = \begin{pmatrix} \mathbf{C} \\ \mathbf{C} \cdot \mathbf{A} \\ \vdots \\ \mathbf{C} \cdot \mathbf{A}^{n-1} \end{pmatrix}, \qquad \mathbf{\Omega}_{1:m} = \begin{pmatrix} x_1 & \mathbf{A} \cdot x_1 & \dots & \mathbf{A}^{m-1} \cdot x_1 \end{pmatrix},$$

where $n = \left[\frac{T}{2}\right]$, $m = T - n + 1$.

Kung's method [1] of extracting of a subspace is based on a singular decomposition of the Hankel matrix

$$\mathbf{H} = \mathbf{U} \cdot \mathbf{S} \cdot \mathbf{V}',$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices and the diagonal matrix $\mathbf{S}$ contains only real positive singular values in the decreasing order which are related to principal components.

One can extract components related to a useful signal. For example let's take the first k principal components:

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_{1:k} & \mathbf{U}_{k+1:n} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{V}_{1:k} & \mathbf{V}_{k+1:n} \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} \mathbf{S}_{1:k} & 0 \\ 0 & \mathbf{S}_{k+1:n} \end{pmatrix}.$$

Then we can restore denoised signal $\bar{\mathbf{H}} = \mathbf{U}_{1:k} \cdot \mathbf{S}_{1:k} \cdot \mathbf{V}'_{1:k}$ and estimate $\mathbf{A}$ via OLS using $\hat{\mathbf{\Gamma}}_{1:n} = \mathbf{U}_{1:k} \cdot \mathbf{S}_{1:k}$:

$$\mathbf{\Gamma}_{1:n-1} \cdot \mathbf{A} = \mathbf{\Gamma}_{2:n} \quad \Rightarrow \quad \hat{\mathbf{A}} = \hat{\mathbf{\Gamma}}^{+}_{1:n-1} \cdot \hat{\mathbf{\Gamma}}_{2:n},$$

with $\hat{\mathbf{\Omega}}_{1:m} = \mathbf{V}'_{1:k}$. this allows us to forecast the trend of future signal for h periods.

$$\bar{y}_{T+h} = \mathbf{\Gamma}_n \cdot \mathbf{A}^h \cdot \mathbf{\Omega}_m.$$

2 Example: GDP at current prices

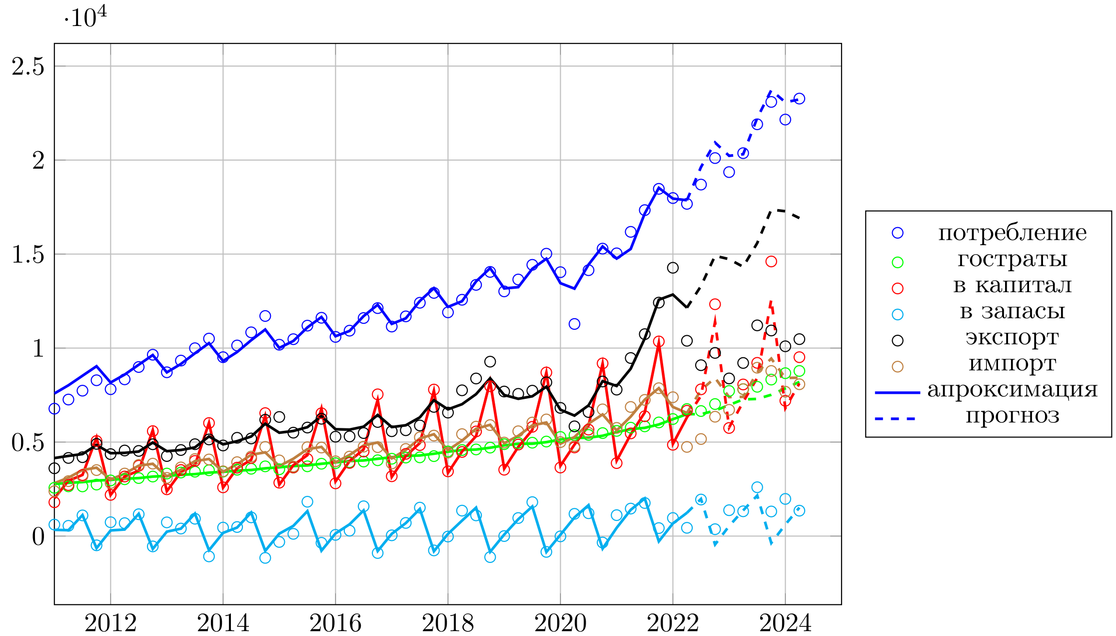

Figure 1: Principal components: 1 2 3 4 5

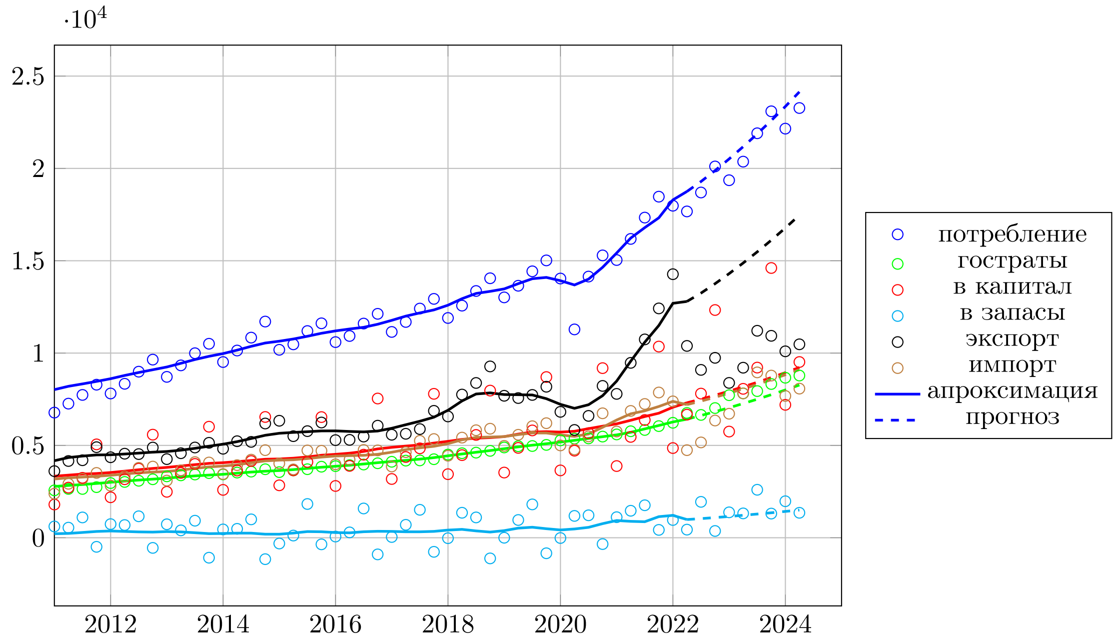

Figure 2: Principal components: 1 5 7 10

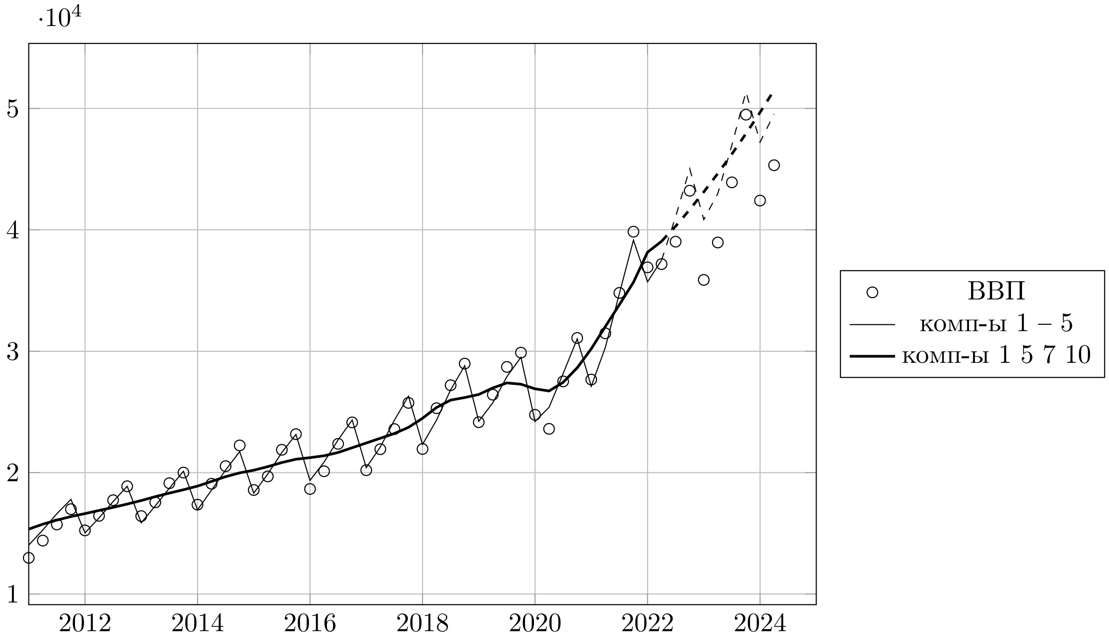

Figure 3: Nominal GDP folded from its components

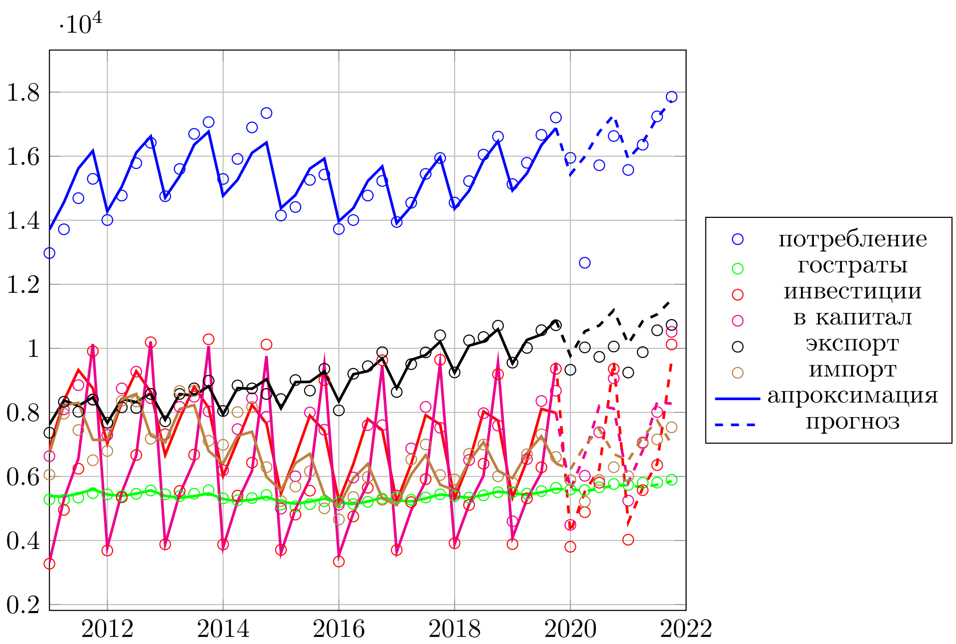

Figure 4: Using of ВВП in prices 2021. Principal components: 1 2 3 4 5

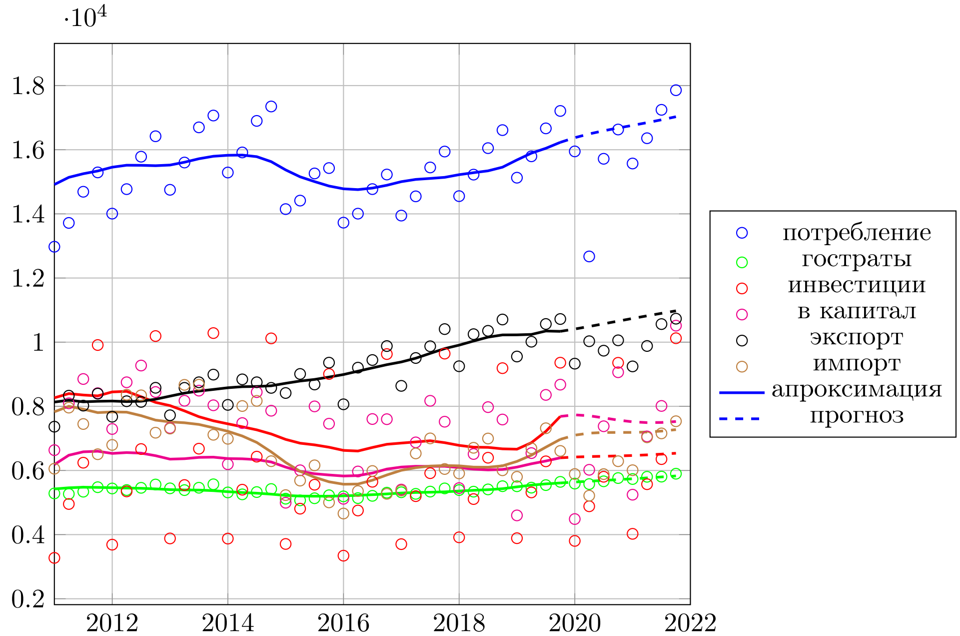

Figure 5: Principal components: 1 5 7 10

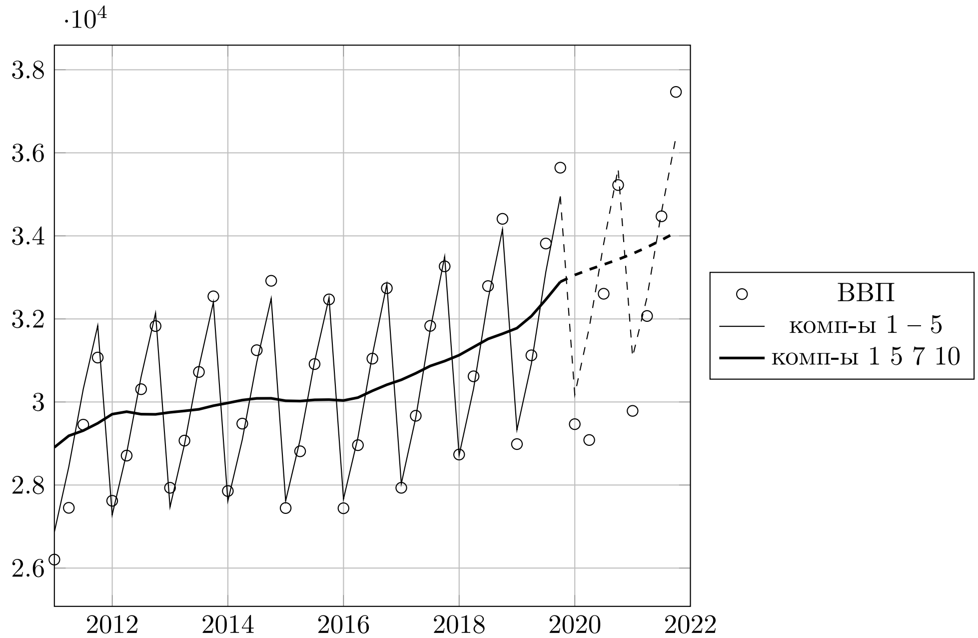

Figure 6: GDP in prices of 2021, folded from its components

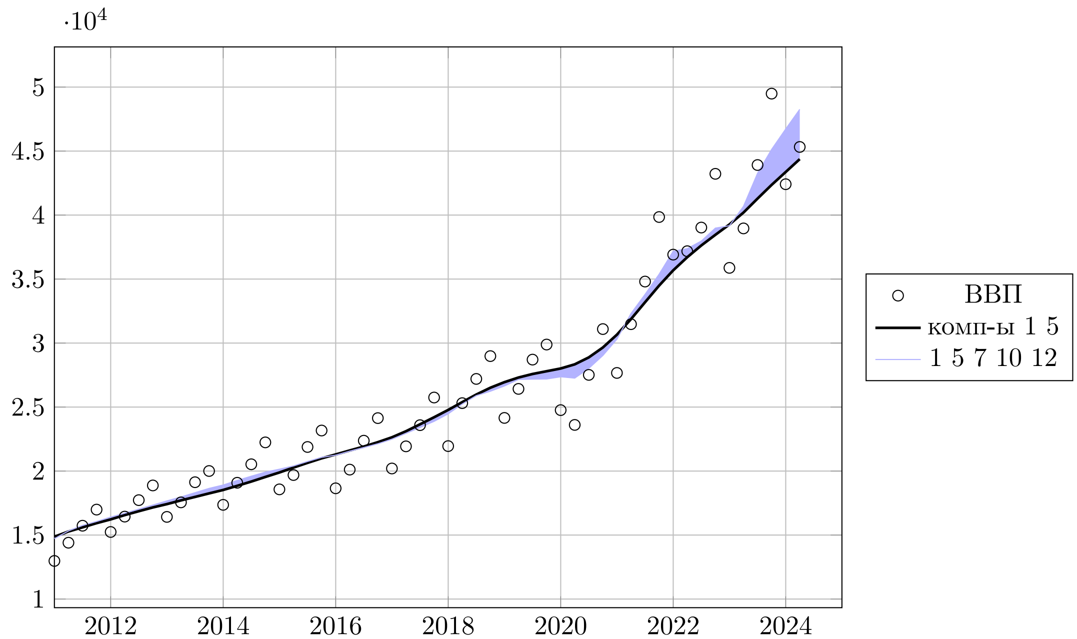

Figure 7: Nominal GDP gap

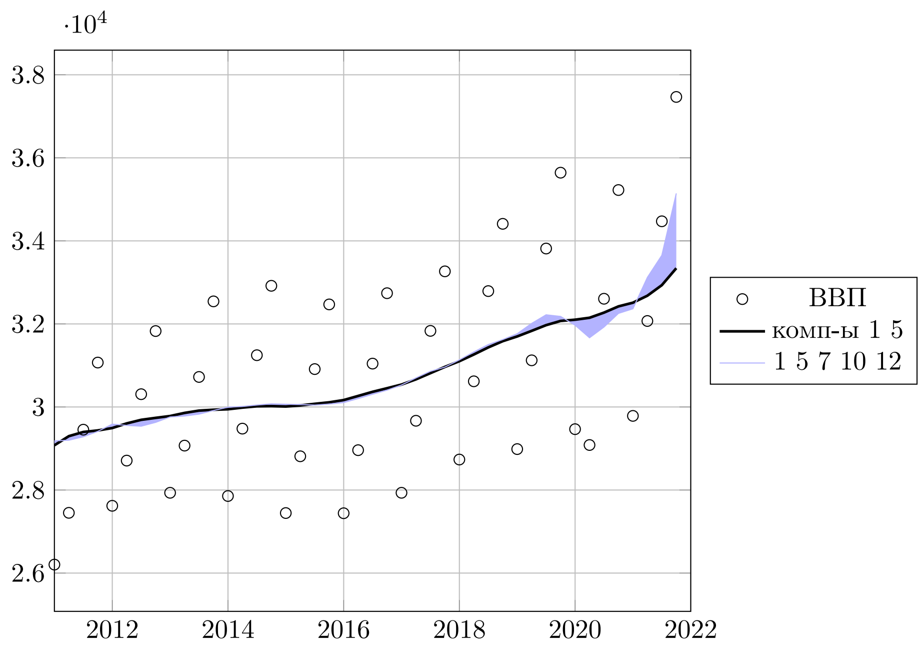

Figure 8: Gap of GDP in prices of 2021

3 Conclusion

1. Using the Kung method of state subspace extraction, it is possible to smooth out a multidimensional series of macro variables, eliminating the main components associated with seasonal characteristics.

2. One can extrapolate the series to the forward periods and add its components to obtain a forecast of GDP.

3. It is possible to estimate the output gap by components that do not include seasonality and trend.

4. Comparison with econometric models shows high predictive properties.

References

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