THE EXISTENCE OF A PURE-STRATEGY EQUILIBRIUM IN A DISCRETE PONDS DILEMMA

VASILY GUSEV*, ALEXANDER NESTEROV†, MIKHAIL RESHETOV¶, AND ALEX SUZDALTSEV§

ABSTRACT. In a variety of economic situations discrete agents choose one resource among several available resources and, once admitted to the resource of choice, divide it among fellow agents admitted there. The amount of the resource an agent gets is proportional to her relative ability to acquire this particular resource, what we refer to as an agent's weight at the resource. The relevant applications include students self-selecting into colleges, politicians self-selecting into races, and athletes self-selecting into teams. We find that this game has a pure-strategy Nash equilibrium in at least three special cases: 1) when agents have the same weight at each resource, 2) when all resources are the same, 3) when there are only two resources. We also show that this game always has an approximate Nash equilibrium when the number of players is large. Existence in the general case remains an open problem.

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1. INTRODUCTION

Consider the admissions game played every year by prospective PhD students when selecting a PhD program. Each school has a certain amount of relevant "resource" that encompasses its faculty time, research supervision, stipends and research funds, and placement efforts. When selecting a school, each student aims at getting the highest amount of

*GAME THEORY LAB, HSE UNIVERSITY, 16, SOYUZA PECHATNIKOV ST., 190121, ST. PETERSBURG, RUSSIAN FEDERATION, vgusev@hse.ru
†DEPARTMENT OF ECONOMICS AND GAME THEORY LAB, HSE UNIVERSITY, 16, SOYUZA PECHATNIKOV ST., 190121, ST. PETERSBURG, RUSSIAN FEDERATION, nesterovu@gmail.com
¶GAME THEORY LAB, HSE UNIVERSITY, 16, SOYUZA PECHATNIKOV ST., 190121, ST. PETERSBURG, RUSSIAN FEDERATION, ma_reshetov@mail.ru , AND DEPARTMENT OF ECONOMICS, LOMONOSOV MOSCOW STATE UNIVERSITY, 1-46, LENINSKIYE GORY, GSP-1, 119991, MOSCOW, RUSSIAN FEDERATION
§DEPARTMENT OF ECONOMICS, HSE UNIVERSITY, 16, SOYUZA PECHATNIKOV ST., 190121, ST. PETERSBURG, RUSSIAN FEDERATION, asuzdaltsev@gmail.com
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this resource to maximize his career opportunities. But whether a student gets more or less of a school's resource depends on who else is admitted at this school and on their relative strength or ranking. Because of that, being the "biggest fish" in a "small pond" might fare better than being a "small fish" in the "biggest pond";[1] whether this is so is the question of the famous *"ponds dilemma"*. Other relevant applications include politicians selecting the most promising constituency for running, firms searching for a new niche to occupy, and athletes choosing a team. In this paper, we formulate and study this game.

One can see this game of choosing schools as an instance of a so-called *congestion game* (Rosenthal, 1973), where the amount of resource a student gets at a school depends on the set of other students admitted to this school. We endow each student with a certain "weight" at each school, and the amount of a school's resource the student gets is proportional to her weight — in relation to all students admitted to this school. The weights may be interpreted as the students' relative advantages or strengths at various schools.

Does this game always have a Nash equilibrium in pure strategies? The answer is not obvious, since finite games, and congestion games in particular, do not necessarily have one. We provide four partially affirmative answers to this question.

(1) If each student has *the same weight at each school*, then an equilibrium can be found using the following greedy algorithm. We consider the students in the weight-descending order and place each student in the school he prefers the most given the placement of the previously admitted students. We show that with this placement, none of the students will prefer to move to a different school (Theorem 1).

(2) If each school has *the same amount of the resource*, we prove the existence of equilibrium by constructing a generalized ordinal potential function of a novel form. *A generalized ordinal potential function* is a function from the set of strategy profiles to reals such that it increases whenever any player switches schools and becomes better-off. The strategy profile maximizing this function is Nash equilibrium. Our potential function is defined, roughly, as follows: for each school, sum up the weights of all students admitted to this school, and then compute the product of all these sums (Theorem 2).

(3) In the general case where schools have possibly different resources and students have possibly different weights at different schools, we prove the existence for the case of *two schools*. We show that starting from any state, a certain tatonnement process does not have cycles and thus converges to an equilibrium (Theorem 3).

(4) Finally, we provide a result for large-scale games. As the number of agents goes up, there exists a strategy profile, which we call an approximate Nash equilibrium,

[1]Indeed, Conley and Önder (2014) show that top students from lesser-ranked PhD programs in Economics often have significantly greater future publishing success than median students from top PhD programs.

such that each agent's incentives to deviate from his strategy vanish, in the sense that any possible gain from deviation is an arbitrarily small fraction of the received payoff. We show the existence of such a profile using a modification of the ordinal potential function from the result 2. This modified function effectively serves as an *approximate generalized ordinal potential* (Theorem 4).

We also discuss whether finding and reaching an equilibrium is easy. In the case 1 (school-independent weights), our algorithm computes the equilibrium in n steps where n is the number of students. Also, at least in cases 1 and 2, the game has *finite improvement property*, that is, any better-reply Nash dynamic converges to an equilibrium. Thus, in these cases, an equilibrium might be reached spontaneously.

Independently and concurrently, this problem was studied by Bilò et al. (2023) under the term "project games". They focus on the computational complexity and efficiency of equilibria, show existence in cases 1 and 2, and call case 3 an interesting open problem (which we resolve).

1.1. **Related literature.** Apart from Bilò et al. (2023), the present paper is related to at least three literatures.

First, as we mentioned above, we study effectively a congestion game, a game of competition for limited resources. This class of games has long attracted the attention of researchers in various disciplines. Rosenthal (1973) pioneered the potential function approach to establishing the existence of a pure-strategy Nash equilibrium in a set of congestion games. Other notable contributions to the study of congestion and potential games were made by Monderer and Shapley (1996), Milchtaich (1996), Harks and Klimm (2012), Harks and Klimm (2015), Kukushkin (2017). The closest paper to ours is Milchtaich (2009) which summarizes the state of knowledge about the existence of pure-strategy Nash equilibria in *separable* weighted congestion games. We further discuss the relation to separable weighted congestion games in section 2.

Second, this paper is related to the literature on sorting that analyzes the aforementioned "ponds dilemma". Damiano et al. (2010) study sorting across organizations where the individuals care about their rank in an organization and the average ability of their peers. They show that sorting and mixing coexist in equilibrium. Morgan et al. (2018) study sorting across contests with varying attractiveness and discriminativeness and find that entry into the big pond is non-monotonic in ability along with other surprising results. Other papers in this literature include Azmat and Möller (2009), Konrad and Kovenock (2012) (for a comprehensive review, see Morgan et al. (2018)). Importantly, in all these studies, the existence of an equilibrium is relatively easy to establish because the authors assume a continuum of players, allow for mixed strategies, and/or introduce substantial symmetry into the problem. In contrast, in this paper, we analyze the case of a finite number of asymmetric players and ask whether a pure-strategy NE exists, in the spirit of the congestion games literature.

Third, and perhaps less obviously, our setting is a special case of matching with externalities. Each pure-strategy Nash equilibrium of our school-choice game corresponds to a pairwise stable many-to-one matching in the sense of Leshno (2022), Pycia and Yenmez (2023) in the special case where schools have non-binding capacity, every student is acceptable to any school and every school is acceptable to any student — in this case, the schools' preferences over students do not play a role. Externalities exist because a student's utility at a school depends on who his classmates are. Sasaki and Toda (1996) first demonstrated that in general, a pairwise-stable matching does not exist if externalities are present. Yet, a stable matching exists under various restrictions on preferences and types of externalities (Mumcu and Saglam (2010), Bando (2012), Pycia and Yenmez (2023)). Our findings (and, in fact, the earlier findings in the congestion games literature) can be viewed as appending this list of important special cases. Leshno (2022) develops a framework for matching with externalities with general peer-dependent preferences and continuum of students. He shows that with a continuum of students, a stable matching exists if a relatively weak "diversity of preferences" condition is satisfied. Also, he shows that an approximately stable matching exists in a large but finite economy with high probability. We show that in our setting an approximate NE exists in a large economy with probability 1 (see subsection 3.4).

2. Model

Let there be a finite set of players $I = (i_1, \dots, i_n)$, which we call "students", and a finite set of locations $S = (s_1, \dots, s_m)$, which we call "schools". These names are for concreteness; as outlined above, other interpretations are also possible. School s has a resource of size r_s; we denote the vector of the schools' resources by $\mathbf{r}$. Student i at each school s has a weight $w_{is} > 0$. We denote the matrix of all weights by W. The weights stem from students' abilities and/or quality of match with a particular school. A tuple $\Gamma = (I, S, \mathbf{r}, W)$ defines an *admissions game*. Each student's i strategy in this game is which school $s_i \in S$ to select. Students select schools simultaneously. Each strategy profile induces a many-to-one matching between students and schools. Abusing notation, we denote by μ both the matching[2] and strategy profile itself. We write $\mu(s)$ to denote the set of students at school s under strategy profile μ. $\mu(s)$ may be empty.

[2]For our purposes, the following definition of a many-to-one matching suffices. A matching μ is a function $S \to 2^I$ such that (i) $\mu(s_1) \cap \mu(s_2) = \emptyset$ for all $s_1 \neq s_2$; (ii) $\bigcup\limits_{s \in S} \mu(s) = I$.

At a strategy profile μ the resource r_s of school s is divided among all students in school s in proportion to their weights: for each $i \in \mu(s)$ the amount of the resource she gets is

$$r_{is}(\mu) = r_s \frac{w_{is}}{\sum_{j\in\mu(s)} w_{js}}. \tag{1}$$

The resources and weights are common knowledge.

Each student selects a school to maximize the amount of the resource she gets. A Nash equilibrium (in pure strategies) is a strategy profile μ such that no student wants to unilaterally change her strategy. By "Nash equilibrium" we always mean Nash equilibrium in pure strategies.

To better understand the mechanics of the game, consider the following two examples.

Example 1. *Suppose $n = 3$, $S = \{A, B\}$, $r_A = 2$, $r_B = 1$, $w_{1A} = w_{1B} = 6$, $w_{2A} = w_{2B} = 2$, $w_{3A} = w_{3B} = 1$. School A is a "big pond" as it has twice as much resource as school B, while school B is a "small pond". According to the weights, student 1 would be a "big fish" in either pond, while students 2 and 3 would be "smaller fish" in either pond.*

It is straightforward to show that in this case the strategy profile μ_0 such that $\mu_0(A) = \{1\}$, $\mu_0(B) = \{2, 3\}$ is a unique pure-strategy NE. In it, student 2 decides to be a big fish in a small pond (school B) rather than a small fish in a big pond (school A), because going to A she would get $2\frac{2}{2+6} = \frac{1}{2}$ units of the resource while going to B she gets $1\frac{2}{2+1} = \frac{2}{3} > \frac{1}{2}$ units of the resource.

Example 2. *Suppose $n = 3$, $S = \{A, B\}$, $r_A = r_B = 1$, $w_{1A} = w_{1B} = 6$, $w_{2A} = w_{3B} = 2$, $w_{3A} = w_{2B} = 1$. The schools are equally resourceful, but the weights are such that student 2 is better suited for school A than for school B while the opposite is true for student 3. (Say, student 2 is more interested in the fields of research in which A is stronger than B and therefore he expects to get a greater share of the resource in A than in B.)*

A strategy profile μ_0 given by $\mu_0(A) = \{1\}$, $\mu_0(B) = \{2, 3\}$ is again a unique NE up to relabelling of schools. In it, student 2 decides to stay in the school B, which he is less suited to, as at A he would get $1\frac{2}{2+6} = \frac{1}{4}$ units of the resource while at B he gets $1\frac{1}{1+2} = \frac{1}{3} > \frac{1}{4}$ units of the resource. The reason is that the school for which he is suited is occupied by a very strong student who takes up most of the school's resource.

Now we are ready to describe the relation to literature in greater detail.

Milchtaich (2009) defines a weighted congestion game with multiplicatively separable preferences as a game where players' costs (to be minimized) are of the form

$$c^i = a_{is} l \left(\sum_{j\in\mu(s)} w_{js} \right) \tag{2}$$

where a_{is} is a coefficient depending on the pair of player i and resource (strategy) s and $l(\cdot)$ is a nondecreasing function. Comparing (2) with the reciprocal of (1), we see that the game we study is a weighted congestion game with multiplicatively separable preferences, a linear cost function $l(w) = w$ and one in which a_{is} are not free parameters but are related to the weights by $a_{is} = \frac{1}{r_s w_{is}}$.

When $a_{is} = \frac{1}{r_s}$, one obtains the *job balancing problem* (see, e.g., Even-Dar et al. (2003) and Milchtaich (2009)) where players are selfish jobs competing for processing by machines and r_s is the speed of processing on machine s. When the weights are school-independent, that is, $w_{is} \equiv w_i$, for all i and s, the game we consider is strategically equivalent to the corresponding job-balancing problem since the w_i factor is a constant from player's i perspective; the game with $a_{is} = \frac{1}{r_s w_i}$. is the same as the one with $a_{is} = \frac{1}{r_s}$. The existence of a pure-strategy Nash equilibrium for the job-balancing problem is a well-known result. Thus, our Theorem 1 below is not new. We include a constructive proof for completeness. The main novelty of our paper, however, lies in the case when the weights w_{is} are not school-independent, and so the game we consider cannot be reduced to a job-balancing problem. All other theorems in this paper pertain to this case. In general, much less is known about the existence of a pure-strategy equilibrium in congestion games when the players' weights (or "demands") are resource-dependent.

Many congestion games not only possess a pure-strategy Nash equilibrium but also have the desirable property that an equilibrium can be reached spontaneously, as any myopic better-reply dynamic always converges. This property is called *finite improvement property* (FIP). Formally, *an improvement path* is a sequence of strategy profiles, each differing from the preceding one only in the strategy of a single player i, such that the change of strategy makes i better-off. A game has the *finite improvement property* if any improvement path is finite. See Monderer and Shapley (1996), Milchtaich (1996), Milchtaich (2009) for the discussion of this property in the context of congestion games. We can establish the FIP for one of our special cases and know from the literature that it holds in another special case. We conjecture that the FIP holds in general for the game studied in this paper.

3. Results

We show that a NE exists in the game given by (1) in three special cases and that an approximate NE exists in the general case when the number of students is large. The proof techniques in the three special cases are substantially different.

3.1. **Arbitrary resources, school-independent weights.**

As discussed above, the existence of a NE in this case follows from previous results. For completeness, we first provide a self-contained, constructive proof of the existence.

Let each student i have the same weight at each school, i.e. $w_{is} \equiv w_i$ for all i and s. We say that in this case, the weights are **school-independent**. This is so in Example 1. Let the students be ordered so that $w_1 \geq w_2 \geq \ldots \geq w_n$. ($w_k$ is the weight of student i_k, $k = 1, \ldots, n$.)

We now prove the existence of a Nash equilibrium in this setting.

Theorem 1. *The admissions game with school-independent weights has a pure-strategy Nash equilibrium.*

Proof. The proof is by induction.

The induction base. Consider an arbitrary (admissions) game $(I, S, \mathbf{r}, W)$ with only 1 student, $I = \{i_1\}$. This game has a Nash equilibrium μ where student i_1 goes to the school with the largest resource and gets the entire resource $r_{is}(\mu) = r_s$.

The induction step. Suppose each game with $|I| \leq n - 1$ students has a Nash equilibrium. We shall prove the same for all games with $|I| = n$ students.

Consider an arbitrary game $(I, S, \mathbf{r}, W)$ with $|I| = n$ and a related smaller game $(I \setminus \{i_n\}, S, \mathbf{r}, W)$ where we eliminate the student with the lowest weight i_n. Due to the induction assumption, there exists a Nash equilibrium μ for the smaller game. We now show that if we add i_n back and send him to the best school given other students' placements according to μ, the resulting strategy profile will be a Nash equilibrium of $(I, S, \mathbf{r}, W)$.

Let us match i_n to some school s that gives him the highest resource given μ: for each other school $s' \neq s$ we have

$$r_s \frac{w_n}{\sum_{i \in \mu(s)} w_i + w_n} \geq r_{s'} \frac{w_n}{\sum_{i \in \mu(s')} w_i + w_n}. \tag{3}$$

By the induction hypothesis, no student wanted to deviate at μ. After we matched i_n to s, the situation at s worsened (the total weight of students there increased) while the situation at other schools did not change. Thus, the students $j \notin \mu(s)$ still do not want to deviate after we added i_n to the game and sent him to s.

It remains to show the key thing: that students $j \in \mu(s)$ also do not want to deviate after we added i_n (even though after we added i_n, the situation at their school worsened).

Suppose, by way of contradiction, that some student $j \in \mu(s)$ wants to deviate to a school s'. That is,

$$r_{s'} \frac{w_j}{\sum_{i \in \mu(s')} w_i + w_j} > r_s \frac{w_j}{\sum_{i \in \mu(s)} w_i + w_n}. \tag{4}$$

We shall show that this contradicts the inequality (3). Indeed, as $w_n \leq w_j$ we get

$$r_{s'} \frac{w_n}{\sum_{i \in \mu(s')} w_i + w_n} \geq r_{s'} \frac{w_n}{\sum_{i \in \mu(s')} w_i + w_j} = \frac{w_n}{w_j} \cdot r_{s'} \frac{w_j}{\sum_{i \in \mu(s')} w_i + w_j}.$$

And from inequality (4) we further get

$$\frac{w_n}{w_j} \cdot r_{s'} \frac{w_j}{\sum_{i\in\mu(s')} w_i + w_j} > \frac{w_n}{w_j} \cdot r_s \frac{w_j}{\sum_{i\in\mu(s)} w_i + w_n} = r_s \frac{w_n}{\sum_{i\in\mu(s)} w_i + w_n}. \tag{5}$$

Combining the two inequalities, we get

$$r_{s'} \frac{w_n}{\sum_{i\in\mu(s')} w_i + w_n} > r_s \frac{w_n}{\sum_{i\in\mu(s)} w_i + w_n},$$

which contradicts inequality (3).

Therefore, after i_n is matched to s, no student wants to deviate, and the resulting strategy profile is a Nash equilibrium. □

As a corollary, we get a simple and fast greedy algorithm to find a Nash equilibrium. Let us add students sequentially in a descending weight order. Each added student is assigned to a school that gives him the highest payoff given the current situation. The resulting strategy profile must be a Nash equilibrium.

The algorithm we provided is similar to the one given by Fotakis et al. (2009) for the job-balancing problem (recall from section 2 that in the case of school-independent weights, it is equivalent to the problem in our paper). An early different proof of Theorem 1 follows from the discussion in paragraph 2 in section 8 and the last paragraph of section 3 in Milchtaich (1996). Theorem 1 also follows from more general results in Harks and Klimm (2012). It is also well-known that the job-balancing game and, more generally, weighted congestion games with separable preferences and a_{is} not depending on i, possess the finite improvement property: the equilibrium will be reached spontaneously as a result of any myopic better-reply dynamic (Even-Dar et al. (2003), Fabrikant et al. (2004), Fotakis et al. (2009), Milchtaich (2009)). Moreover, Theorem 4.1 in Kukushkin (2017) ensures the existence of a *strong* Nash equilibrium, i.e., an equilibrium robust to coalitional deviations, in this setting.

One can reduce our game to one of the previously addressed cases only in the case of school-independent weights. In general, it is known much less about the existence of equilibria in congestion games where players' weights are resource-dependent. We prove several results for this case in the following subsections.

3.2. **School-independent resources, arbitrary weights.** In this subsection, we consider the special case opposite to that in the previous subsection. We suppose that each school has the same amount of resources $r_s \equiv r$ but students might have different weights at different schools[3]. Example 2 falls in this category.

[3]This does not belong to the case of "resource-independent costs" defined in Milchtaich (2009) as here the coefficient $a_{is} = 1/(rw_{is})$ still depends on the school s.

We prove the result by constructing a novel generalized ordinal potential function for the game in question. Recall from Monderer and Shapley (1996) that a function P from the set of strategy profiles μ to reals is called a *generalized ordinal potential* of a game if for any player i and two strategy profiles μ' and μ differing only in the position of player i,

$$U_i(\mu') > U_i(\mu) \Longrightarrow P(\mu') > P(\mu) \tag{6}$$

Finite games admitting a generalized ordinal potential clearly possess a pure-strategy Nash equilibrium — a profile μ^* that maximizes $P(\mu)$. Moreover, Monderer and Shapley (1996) and Milchtaich (1996) show that for finite games, having a generalized ordinal potential is equivalent to the finite improvement property (FIP), so the equilibrium may be reached by a myopic better-reply dynamic.

The generalized potential function we construct is defined by

$$P_\varepsilon(\mu) = \prod_{s\in S} \max\left\{\sum_{j\in\mu(s)} w_{js}, \varepsilon\right\}. \tag{7}$$

where ε is a sufficiently small positive number. The version of this function without the ε amendment, i.e.

$$P(\mu) = \prod_{s\in S}\sum_{j\in\mu(s)} w_{js},$$

would have the potential property (6) only for the profiles μ such that all schools have at least one student, as $P(\mu)$ would be zero for all profiles where some school is empty. The ε amendment allows us to take care of the profiles μ with empty schools[4]. ε needs to be sufficiently small so that (i) whenever a school is non-empty, ε does not affect the school's contribution to the potential; (ii) maximizing the potential would lead to making as many schools non-empty as possible. These correspond to the following inequalities:

$$\varepsilon < \min_{i,s} w_{is}, \tag{8}$$

and for every student i, pair of schools $s_1 \neq s_2$, and nonempty set of students $T \subseteq I \setminus \{i\}$,

$$\varepsilon < \frac{w_{is_2} \cdot \sum_{j\in T} w_{js_1}}{w_{is_1} + \sum_{j\in T} w_{js_1}}. \tag{9}$$

Theorem 2. *Take any $\varepsilon > 0$ satisfying (8) and (9). Then, the function $P_\varepsilon(\mu)$ given by (7) is a generalized ordinal potential for the admissions game with school-independent resources.*

Proof. We need to show (6). Consider any two strategy profiles μ' and μ differing only in the position of student i. Suppose i is at school s_1 at μ and is at school $s_2 \neq s_1$ at μ'. That

[4]We are grateful to an anonymous referee for this suggestion.

is, μ' is obtained from μ by transferring i from s_1 to s_2. Suppose $U_i(\mu') > U_i(\mu)$. Consider 4 cases:

(i) $\mu(s_1) = \{i\}$, $\mu'(s_2) = \{i\}$. That is, i is alone at a school under both μ and μ'.
(ii) $\mu(s_1) = \{i\}$, $\mu'(s_2)\setminus\{i\} \neq \emptyset$. That is, i is alone at a school under μ but not under μ'.
(iii) $\mu(s_1)\setminus\{i\} \neq \emptyset$, $\mu'(s_2) = \{i\}$. That is, i is alone at a school under μ' but not under μ.
(iv) $\mu(s_1) \setminus \{i\} \neq \emptyset$, $\mu'(s_2) \setminus \{i\} \neq \emptyset$ (i is not alone in either situation).

As each school has the same amount of resources $r_s \equiv r$, cases (i) and (ii) are incompatible with $U_i(\mu') > U_i(\mu)$. Indeed, in case (i) $U_i(\mu') = r = U_i(\mu)$ while in case (ii) $U_i(\mu') < r = U_i(\mu)$. So we are left with cases (iii) and (iv).

Consider case (iii). In contrast to case (ii), in this case, we always have $U_i(\mu') > U_i(\mu)$, as $U_i(\mu') = r > U_i(\mu)$. We need to show that $P_\varepsilon(\mu') > P_\varepsilon(\mu)$ as well, that is,

$$\prod_{s\in S} \max\left\{\sum_{j\in\mu'(s)} w_{js}, \varepsilon\right\} > \prod_{s\in S} \max\left\{\sum_{j\in\mu(s)} w_{js}, \varepsilon\right\}.$$

Dividing this by the (positive) factors not pertaining to schools s_1, s_2, we get

$$w_{is_2} \sum_{j\in\mu'(s_1)} w_{js_1} > \varepsilon \sum_{j\in\mu(s_1)} w_{js_1}, \tag{10}$$

where we used the conditions in case (iii) and (8) to remove the maximum operators. Clearly, $\sum_{j\in\mu(s_1)} w_{js_1} = w_{is_1} + \sum_{j\in\mu'(s_1)} w_{js_1}$, so (10) may be rewritten as

$$\varepsilon < \frac{w_{is_2} \cdot \sum_{j\in\mu'(s_1)} w_{js_1}}{w_{is_1} + \sum_{j\in\mu'(s_1)} w_{js_1}}.$$

The last inequality is a special case of (9), with $T = \mu'(s_1) = \mu(s_1) \setminus \{i\}$, so it holds by the assumption on ε, and thus $P_\varepsilon(\mu') > P_\varepsilon(\mu)$.

The remaining case (iv) constitutes the heart of this proof. We shall show that within case (iv) we actually have the equivalence of $P_\varepsilon(\mu') > P_\varepsilon(\mu)$ and $U_i(\mu') > U_i(\mu)$. The equivalence follows from the following sequence of equivalent statements:

$$P_\varepsilon(\mu') > P_\varepsilon(\mu)$$

$$\prod_{s\in S} \max\left\{\sum_{j\in\mu'(s)} w_{js}, \varepsilon\right\} > \prod_{s\in S} \max\left\{\sum_{j\in\mu(s)} w_{js}, \varepsilon\right\}$$

$$\sum_{j\in\mu'(s_2)} w_{js_2} \sum_{j\in\mu'(s_1)} w_{js_1} > \sum_{j\in\mu(s_2)} w_{js_2} \sum_{j\in\mu(s_1)} w_{js_1}$$

$$\frac{\sum_{j\in\mu'(s_1)} w_{js_1}}{\sum_{j\in\mu(s_1)} w_{js_1}} > \frac{\sum_{j\in\mu(s_2)} w_{js_2}}{\sum_{j\in\mu'(s_2)} w_{js_2}}$$

$$
\begin{aligned}
\frac{\sum_{j\in\mu'(s_1)} w_{js_1}}{w_{is_1}+\sum_{j\in\mu'(s_1)} w_{js_1}} &> \frac{\sum_{j\in\mu(s_2)} w_{js_2}}{w_{is_2}+\sum_{j\in\mu(s_2)} w_{js_2}}\\
1-\frac{w_{is_1}}{w_{is_1}+\sum_{j\in\mu'(s_1)} w_{js_1}} &> 1-\frac{w_{is_2}}{w_{is_2}+\sum_{j\in\mu(s_2)} w_{js_2}}\\
\frac{w_{is_2}}{w_{is_2}+\sum_{j\in\mu(s_2)} w_{js_2}} &> \frac{w_{is_1}}{w_{is_1}+\sum_{j\in\mu'(s_1)} w_{js_1}}\\
r\frac{w_{is_2}}{\sum_{j\in\mu'(s_2)} w_{js_2}} &> r\frac{w_{is_1}}{\sum_{j\in\mu(s_1)} w_{js_1}}\\
U_i(\mu') &> U_i(\mu),
\end{aligned}
$$

where we have used the conditions in case (iv) and (8) to remove the maximum operators and to be able to divide both parts by $\sum_{j\in\mu(s_1)} w_{js_1}$, $\sum_{j\in\mu'(s_2)} w_{js_2}$. □

As we discussed above, the existence of a generalized ordinal potential implies, in a finite game, the existence of a pure-strategy Nash equilibrium. Indeed, as property (6) may be rewritten as its contrapositive — for all strategy profiles μ and μ' differing only in the position of player i we have $P(\mu) \geq P(\mu') \implies U_i(\mu) \geq U_i(\mu')$ — the strategy profile μ^* maximizing $P(\mu)$ is a Nash equilibrium of the game.

Corollary 1. *The admissions game with school-independent resources has a pure-strategy Nash equilibrium.*

As discussed above, in finite games the existence of a generalized ordinal potential is equivalent to the finite improvement property (Monderer and Shapley, 1996; Milchtaich, 1996), so we can state a further corollary of Theorem 2.

Corollary 2. *The admissions game with school-independent resources possesses the finite improvement property.*

Thus, when the students' weights are possibly school-specific but the schools' resources are equal, a Nash equilibrium of the game can be reached spontaneously, as in the known case of school-independent weights and arbitrary resources that we discussed in section 3.1.

In general, the equivalence of $U_i(\mu') > U_i(\mu)$ and $P(\mu') > P(\mu)$ for some function $P(\mu)$ would mean that $P(\mu)$ is an *ordinal potential* for the game in question (Monderer and Shapley, 1996). However, here the equivalence of $U_i(\mu') > U_i(\mu)$ and $P_\varepsilon(\mu') > P_\varepsilon(\mu)$ does not hold outside case (iv). For example, a student may move from a school where he has a small weight to a school where he has a large weight while being alone at both schools. In this case, $P_\varepsilon(\mu')$ will strictly increase but the student's utility won't change, as resources are school-independent. Thus, $P_\varepsilon(\mu)$, while being a *generalized* ordinal potential of the game, is not its ordinal potential.

Whether a generalized ordinal potential exists in the general case of arbitrary resources and weights remains an open problem. In subsection 3.4, we show that an approximate

NE exists employing a natural generalization of potential (7) that serves effectively as an *approximate generalized ordinal potential* of the game in a certain sense.

3.3. **Arbitrary resources, arbitrary weights: two schools.** In this subsection, we show that a pure-strategy Nash equilibrium exists in the admissions game when there are two schools (and arbitrary student weights and resources). The proof involves showing that a certain natural "controlled tatonnement" process, in which all students willing to move do so in a certain order, cannot cycle, and therefore has to stop.

In principle, a cycle of an algorithm can have a complex structure, and proving the impossibility of cycles case-by-case seems to be infeasible. We gain traction by looking at certain substructures of a cycle and then showing that a smallest such substructure has so many properties that they become incompatible, leading to a contradiction. Still, this is the most involved proof of the present paper[5].

Theorem 3. *The admissions game with two schools and arbitrary resources has a pure-strategy Nash equilibrium.*

Proof. Name the two schools A and B. We first prove the existence of equilibrium for the case where the weights are generic in the sense that for any two distinct students i and j, the ratios w_{iA}/w_{iB} and w_{jA}/w_{jB} are distinct. We then show the existence of equilibrium in general using a limiting argument, approaching a game with possibly non-generic weights with a sequence of games with generic weights.

Suppose the weights are generic. We say that the situation at school s improves (worsens) if the total weight of students at s decreases (increases). Consider the following "controlled tatonnement" process:

(1) Start at an arbitrary strategy profile.
(2) Look at all students at school A who want to move to school B. Transfer to B the student with the highest ratio w_{iB}/w_{iA} among those who want to move.
(3) Repeat (2) until no one wants to move to B.
(4) Then look at all students at school B who want to move to school A. Transfer to A the student with the highest ratio w_{iA}/w_{iB} among those who want to move.
(5) Repeat (4) until no one wants to move to A.
(6) Repeat (2)-(5) until no one wants to move.

We shall show that this process must stop. The strategy profile at which this process stops is by construction such that no one wants to move, i.e., an equilibrium.

[5]Many attempts to construct an equilibrium in a simpler way are not successful. For example, the natural idea that there should be an equilibrium where a student i goes to A if and only if the weight ratio w_{iA}/w_{iB} is sufficiently high, is refuted by Example 2 above in which no such equilibrium exists.

Suppose, by way of contradiction, that this process does not stop. As there are only finitely many strategy profiles, the process must enter a cycle. Consider all sequences L of consecutive moves in this cycle such that:

(1) $|L| > 1$.
(2) The first move and the last move in L are both from school $s_1 \in \{A, B\}$ to school $s_2 \neq s_1$.
(3) The student making the first move in L has the highest ratio w_{is_1}/w_{is_2} among all students moving in L.
(4) The student making the last move in L is at s_1 during L (before his move).
(5) All students making intermediate moves in L are at s_2 just after the first move in L.

Let $\mathcal{L}$ be the set $\mathcal{L}$ of all such sequences. We first show $\mathcal{L}$ is non-empty by pinpointing a sequence $L_0 \in \mathcal{L}$. Consider the student f who has the highest ratio w_{iA}/w_{iB} among all students moving in the cycle. Because it is a cycle, student f has to move from A to B at a certain step. We start L_0 with this move of f. f also has to willingly move from B back to A at some further step. Now note that there must be a student who is at A after the f's move from A to B and who moves from A to B before f moves from B to A. If there were no such student, the situation at B would weakly improve and at A weakly worsen after f's move from A to B, but this means that f would not change her mind and move from B back to A. Therefore, such a student exists; of all such students, take the one who moves from A to B first after f. Call her l. We end L_0 with this move of l. Now, $L_0 \in \mathcal{L}$ because:

(1) $|L_0| > 1$ by construction (L_0 includes at least the moves of f and l).
(2) The first move and the last move in L_0 are both from school A to school B.
(3) f has the highest ratio w_{iA}/w_{iB} among all students moving in L_0 as she has the highest ratio w_{iA}/w_{iB} among all students moving in the cycle.
(4) The student making the last move in L_0, l, is at A during L_0 (before his move) by construction.
(5) All students making intermediate moves in L_0 are at B just after the first move in L_0 because l is the first student to move out of those who are at A after the first move.

Now, as we know that $\mathcal{L} \neq \emptyset$, we can consider a sequence $L_{\min} \in \mathcal{L}$ with the smallest number of moves (smallest length). With a slight abuse of notation, call the first moving student in $L_{\min}$ again f and the last moving student in $L_{\min}$ again l. Without loss of generality, let f and l move from A to B.

By construction, l is at A just before f moves. If l wanted to go to B at that moment, she would be transferred by our algorithm to B *before* f, since by construction $w_{lA}/w_{lB} <$

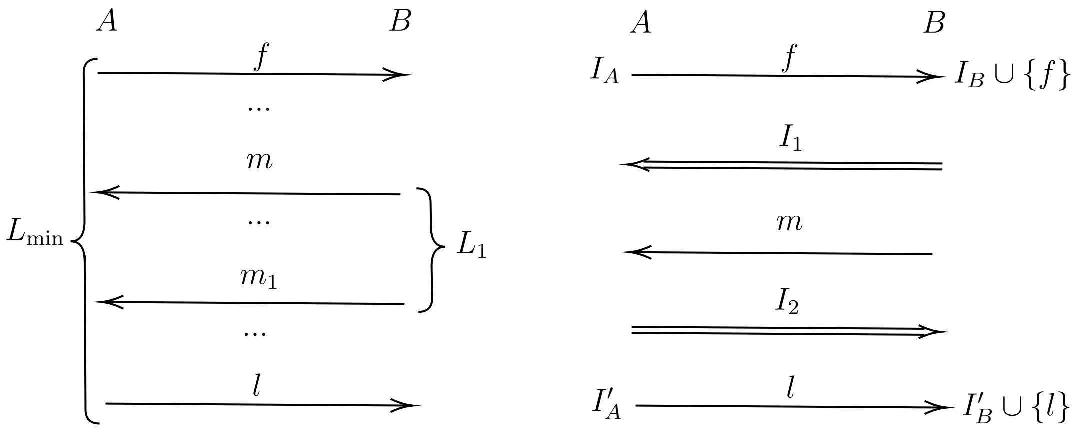

Figure 1. Left: The existence of a student m_1 moving from B to A after m contradicts the minimality of $L_{\min}$. Right: The only possible scheme of moves in $L_{\min}$. The sets I_1 and I_2 are the sets of students moving *eventually* in the respective directions (the movements of students switching schools back and forth in between f's and m's moves and in between m's and l's moves are not shown).

w_{fA}/w_{fB} so $w_{lB}/w_{lA} > w_{fB}/w_{fA}$. However, our algorithm did not transfer her, so it must be that she does not want to go to B at that moment. However, she does want to go to B at the end of $L_{\min}$, so she changes her mind. Thus, either the situation in A must have worsened or the situation in B must have improved during $L_{\min}$ (or both). Let $w_s(J)$ be the total weight of a set of students J at school s. Denoting by I_A, I_B the sets of students at A and B just before f moves and by I'_A, I'_B the sets of students at A and B just before l moves we get:

$$\text{At least one of the inequalities } w_A(I'_A) > w_A(I_A),\ w_B(I'_B) < w_B(I_B) \text{ holds.} \tag{11}$$

Using the minimality of $L_{\min}$ and some subtle quantitative analysis we shall eventually show that in fact both inequalities in (11) are violated, thus arriving at a contradiction.

Because of (11), there must be a student moving in between f and l in $L_{\min}$, that is, $|L_{\min}| > 2$. (Otherwise, we would have $w_A(I'_A) = w_A(I_A) - w_{fA} \le w_A(I_A)$ and $w_B(I'_B) = w_B(I_B) + w_{fB} \ge w_B(I_B)$, contradicting (11).) Consider all moves in $L_{\min}$ except the first and the last, and call the set of students involved in these intermediate moves I^{int}. By property (5) of $L_{\min}$, any student $j \in I^{int}$ is at B after the f's move. Now consider the student with the *minimal* ratio w_{iA}/w_{iB} across all students in I^{int}, call her m. As $m \in I^{int}$, she is at B after f's move, so m's first move in $L_{\min}$ is from B to A.

We shall show that there is no student who is at B just after m's first move and who moves from B to A at some point after m in $L_{\min}$. Suppose there was such a student; of all such students, let m_1 be a student that moves first in $L_{\min}$ after m.

Then consider a sequence of moves L_1 that includes all moves starting with the m's move and ending with the m_1's move (inclusive). (See Figure 1, left.) Now we check that L_1

satisfies all the 5 properties for the inclusion in $\mathcal{L}$ (with $s_1 = B$). (1) $|L_1| > 1$ as it includes at least the moves of m and m_1; (2) The first and last moves in L_1 are both from B to A; (3) m has the highest ratio w_{iB}/w_{iA} among all students moving in L_1 as she has the lowest ratio w_{iA}/w_{iB} among all students in I^{int} by construction; (4) m_1 is at B during L_1 by construction; (5) all students making intermediate moves in L_1 are at A just after m's move as m_1 is the first student moving in $L_{\min}$ after m among those who are at B just after m's move.

Thus, $L_1 \in \mathcal{L}$. However, L_1 is shorter than $L_{\min}$, contradicting the minimality of the latter. Thus, no such student m_1 exists. In other words, all students who are at B just after m's move are also in I'_B: no student who is at B just after m's move eventually switches school from B to A during $L_{\min}$.

Denote by I_1 the set of students eventually changing school in between f's and m's moves. By property 5 of $L_{\min}$ all the students in I_1 are at B right after f's move so all of them change school from B to A. Denote by I_2 the set of students eventually changing school in between m's and l's moves. By the analysis in the previous paragraph, all the students in I_2 are at A after m's move and so all of them change school from A to B. The resulting scheme of moves in $L_{\min}$ is shown in Figure 1, right.

Given this,

$$w_A(I'_A) = w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA} - w_A(I_2) \tag{12}$$

and

$$w_B(I'_B) = w_B(I_B) + w_{fB} - w_B(I_1) - w_{mB} + w_B(I_2). \tag{13}$$

In the remaining part of the proof, we shall show that $w_{fA} > w_A(I_1) + w_{mA}$ and $w_{fB} > w_B(I_1) + w_{mB}$. Given (12), (13), $w_A(I_2) \geq 0$ and $w_B(I_2) \geq 0$, these will imply that $w_A(I'_A) < w_A(I_A)$ and $w_B(I'_B) > w_B(I_B)$, thus contradicting (11).

As f wants to move,

$$r_A \frac{w_{fA}}{w_A(I_A)} < r_B \frac{w_{fB}}{w_B(I_B) + w_{fB}}.$$

As m wants to move,

$$r_A \frac{w_{mA}}{w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA}} > r_B \frac{w_{mB}}{w_B(I_B) + w_{fB} - w_B(I_1)}.$$

Isolating r_B/r_A from both inequalities, we get

$$\frac{w_{mA}}{w_{mB}} \frac{w_B(I_B) + w_{fB} - w_B(I_1)}{w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA}} > \frac{w_{fA}}{w_{fB}} \frac{w_B(I_B) + w_{fB}}{w_A(I_A)}.$$

Given that $w_B(I_1) \geq 0$, after cancelling $w_B(I_B) + w_{fB}$ and rearrangement we get:

$$\frac{w_{mA}}{w_{mB}} w_A(I_A) > \frac{w_{fA}}{w_{fB}} \left(w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA} \right). \tag{14}$$

As by construction $\frac{w_{fA}}{w_{fB}} > \frac{w_{mA}}{w_{mB}}$, the RHS of (14) is in its turn larger than

$$\frac{w_{mA}}{w_{mB}}\left(w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA}\right).$$

Thus,

$$\frac{w_{mA}}{w_{mB}} w_A(I_A) > \frac{w_{mA}}{w_{mB}}\left(w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA}\right).$$

After rearrangement, we get

$$w_{fA} > w_A(I_1) + w_{mA}. \tag{15}$$

Now we show that an analogous inequality holds for the weights in B. Using $\frac{w_{fA}}{w_{fB}} > \frac{w_{mA}}{w_{mB}}$ again we can obtain a sharper lower bound for the RHS of (14):

$$\frac{w_{fA}}{w_{fB}}\left(w_A(I_A) - w_{fA} + w_A(I_1) + w_{mA}\right) > \frac{w_{mA}}{w_{mB}}\left(w_A(I_A) - w_{fA}\right) + \frac{w_{fA}}{w_{fB}}\left(w_A(I_1) + w_{mA}\right).$$

Combining this with (14), we get

$$\frac{w_{mA}}{w_{mB}} w_A(I_A) > \frac{w_{mA}}{w_{mB}}\left(w_A(I_A) - w_{fA}\right) + \frac{w_{fA}}{w_{fB}}\left(w_A(I_1) + w_{mA}\right),$$

so

$$\frac{w_{mA}}{w_{mB}} w_{fA} > \frac{w_{fA}}{w_{fB}}\left(w_A(I_1) + w_{mA}\right).$$

After cancellations and rearrangement, this becomes

$$w_{fB} > \frac{w_{mB}}{w_{mA}} w_A(I_1) + w_{mB}. \tag{16}$$

Now we argue that $\frac{w_{mB}}{w_{mA}} w_A(I_1) > w_B(I_1)$. This inequality can be rewritten as $\frac{w_{mB}}{w_{mA}} > \frac{w_B(I_1)}{w_A(I_1)}$. Note that $\frac{w_B(I_1)}{w_A(I_1)}$ is a weighted average of ratios $\frac{w_{iB}}{w_{iA}}$ for all $i \in I_1$. However, by construction $\frac{w_{mB}}{w_{mA}} > \frac{w_{iB}}{w_{iA}}$ for all $i \in I_1$. Thus, $\frac{w_{mB}}{w_{mA}} > \frac{w_B(I_1)}{w_A(I_1)}$ as well. Therefore, $\frac{w_{mB}}{w_{mA}} w_A(I_1) > w_B(I_1)$. Combining this with (16), we get

$$w_{fB} > w_B(I_1) + w_{mB}. \tag{17}$$

Now note that (15) and (12) together with $w_A(I_2) \geq 0$ imply that $w_A(I'_A) < w_A(I_A)$. Likewise, (17) and (13) together with $w_B(I_2) \geq 0$ imply that $w_B(I'_B) > w_B(I_B)$. The inequalities $w_A(I'_A) < w_A(I_A)$ and $w_B(I'_B) > w_B(I_B)$ together contradict (11).

Now suppose we have a game Γ_0 with non-generic weights W_0. Consider a sequence of games Γ^k with the same vector of resources $\mathbf{r}$ and generic weights W^k, $k = 1, 2, \ldots$, such that W^k converges to W_0. By the argument above, each game Γ^k has at least one pure-strategy Nash equilibrium. Denote the set of pure-strategy NE of Γ^k by $NE(\Gamma^k) \neq \emptyset$. As there is only a finite number of strategy profiles, there exists a strategy profile μ_0 such that $\mu_0 \in NE(\Gamma^k)$ for infinitely many k. This means that a certain set of weak inequalities holds at μ_0 for infinitely many k. As payoffs are continuous in W and weak inequalities are preserved when taking a limit, the inequalities must hold also at the limit along this subsequence, i.e. with weights W_0. Thus, $\mu_0 \in NE(\Gamma_0)$ as well, so $NE(\Gamma_0) \neq \emptyset$[6]. $\square$

Note that even though we proved the absence of cycles in our algorithm, it does not imply the finite improvement property. Indeed, we have shown the absence of cycles in *some* better-reply dynamic but not in any such dynamic.

3.4. **Arbitrary resources, arbitrary weights: an asymptotic result.** In this subsection, we consider the general case of arbitrary resources, arbitrary weights, and an arbitrary number of schools. In this case, we establish the existence of an approximate pure-strategy Nash equilibrium when the number of students is sufficiently large. We do this by constructing a natural extension of the generalized ordinal potential (7) for the case of school-specific resources. Even though this function is not a generalized ordinal potential of the game, it will effectively serve as its *approximate generalized ordinal potential.*

We first describe the main idea informally and then provide the formal result with all the details filled in.

Consider the following function from the set of strategy profiles to real numbers:

$$\tilde{P}(\mu) := \prod_{s \in S} \left(\sum_{j \in \mu(s)} w_{js} \right)^{r_s}. \tag{18}$$

This function has a familiar Cobb-Douglas form. The idea behind this generalization of the function (7) is that a school with a resource of 2 should be treated as two of its smaller copies with a resource of 1 each.

Consider a strategy profile μ^* that maximizes $\tilde{P}(\mu)$ and another strategy profile μ' that differs from μ^* only in the position of student i. By optimality of μ^*, $\tilde{P}(\mu^*) \geq \tilde{P}(\mu')$, so

$$\prod_{s \in S} \left(\sum_{j \in \mu^*(s)} w_{js} \right)^{r_s} \geq \prod_{s \in S} \left(\sum_{j \in \mu'(s)} w_{js} \right)^{r_s}. \tag{19}$$

Suppose i is at school s_1 at μ^* and at school s_2 at μ'. Assuming for a moment that μ^* is such that all schools have at least one student (this will follow from optimality of μ^* for n not less than the number of schools), by the same transformations as in the proof of Theorem 2, we obtain from (19)

$$\begin{gathered} \left(\frac{\sum_{j \in \mu^*(s_2)} w_{js_2}}{\sum_{j \in \mu'(s_2)} w_{js_2}} \right)^{r_{s_2}} \geq \left(\frac{\sum_{j \in \mu'(s_1)} w_{js_1}}{\sum_{j \in \mu^*(s_1)} w_{js_1}} \right)^{r_{s_1}} \\ \left(1 - \frac{w_{is_2}}{\sum_{j \in \mu'(s_2)} w_{js_2}} \right)^{r_{s_2}} \geq \left(1 - \frac{w_{is_1}}{\sum_{j \in \mu^*(s_1)} w_{js_1}} \right)^{r_{s_1}}. \end{gathered} \tag{20}$$

[6]We are grateful to an anonymous referee for suggesting this limiting argument.

Now the key idea is that with many students, the share of resources every student is getting will tend to be low. Thus one may consider a Taylor approximation of the LHS and RHS of (20) when these shares are small. That is, for every school s

$$(1-x)^{r_s} = 1 - r_s x + o(x)$$

as $x \to 0$. Applying this to both the LHS and RHS of (20), one gets

$$1 - r_{s_2}\frac{w_{is_2}}{\sum_{j\in\mu'(s_2)} w_{js_2}} + o\left(\frac{w_{is_2}}{\sum_{j\in\mu'(s_2)} w_{js_2}}\right) \geq 1 - r_{s_1}\frac{w_{is_1}}{\sum_{j\in\mu^*(s_1)} w_{js_1}} + o\left(\frac{w_{is_2}}{\sum_{j\in\mu^*(s_2)} w_{js_2}}\right)$$

$$r_{s_1}\frac{w_{is_1}}{\sum_{j\in\mu^*(s_1)} w_{js_1}} \geq r_{s_2}\frac{w_{is_2}}{\sum_{j\in\mu'(s_2)} w_{js_2}} + o\left(\frac{w_{is_2}}{\sum_{j\in\mu^*(s_2)} w_{js_2}}\right) - o\left(\frac{w_{is_2}}{\sum_{j\in\mu'(s_2)} w_{js_2}}\right). \quad (21)$$

Note that, aside from the “small-o” terms, we obtained the expressions corresponding exactly to the student's i utilities at μ^* and μ'! Thus, equation (21) says that if student i deviates from the school prescribed by μ^* her utility will decrease modulo a small remainder. This suggests that μ^* is in fact an approximate pure-strategy Nash equilibrium of the admissions game and the function $\tilde{P}(\mu)$ is, in a sense, its approximate generalized ordinal potential.

Now we establish our formal result.

An approximate Nash equilibrium may be either with an additive or a relative (multiplicative) error ε. The existence of an approximate Nash equilibrium with a fixed *additive* error with a large number of students is almost immediate in our setting. Indeed, any strategy profile with a sufficiently large number of students *at every school* will leave all the students with sufficiently small utility so that the difference of utilities upon deviation will also be small; any such strategy profile will be an approximate Nash equilibrium with additive error. However, with the approximate potential function approach described above we can establish the existence of an approximate Nash equilibrium with small *relative* error, which is much less obvious.

Definition 1. *Suppose player's i utility function in a certain game is $u_i(s_i, s_{-i})$ where s_i is player's i strategy and s_{-i} is the vector of other players' strategies. A profile of strategies s is* an approximate Nash equilibrium with relative error at most $\varepsilon > 0$ *iff for any i and deviation s_i' we have*

$$(1+\varepsilon)u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}). \quad (22)$$

That is, by deviating from such an approximate Nash equilibrium any student can increase her utility by at most a factor of $(1+\varepsilon)$.

We state the existence result under the additional mild assumption that the weights stay uniformly bounded away from zero and bounded from above when the number of students grows to infinity.

Theorem 4. *Fix the set of schools and the vector of resources* $\mathbf{r}$. *Consider a sequence of sets of students* $\{I_n\}_{n=1}^{\infty}$ *with* $|I_n| = n$ *and a sequence of weight matrices* $\{W_n\}_{n=1}^{\infty}$ *such that* W_n *is of size* $n \times |S|$ *and for all* i, s *and* n *each element of* W_n *lies within* $[\underline{w}, \overline{w}]$ *where* $0 < \underline{w} < \overline{w}$. *For any such sequence and any* $\varepsilon > 0$ *there exists* N *such that for all* $n > N$ *the admissions game* $\Gamma_n = (I_n, S, \mathbf{r}, W_n)$ *has an approximate Nash equilibrium with relative error at most* ε.

Proof. Consider, for every n, the strategy profile μ_n^* that maximizes the approximate potential (18). We claim that this strategy profile constitutes an approximate Nash equilibrium for a sufficiently large n.

Let $n \geq |S|$. This implies that μ_n^* is such that there is at least one student at every school (otherwise $\tilde{P} = 0$ and so μ_n^* is surely not optimal). Consider a student i deviating from school s_i at μ_n^* to a school s_i', forming a strategy profile μ'. By the optimality of μ_n^*, $\tilde{P}(\mu_n^*) \geq \tilde{P}(\mu')$, so

$$\prod_{s \in S} \left(\sum_{j \in \mu^*(s)} w_{js} \right)^{r_s} \geq \prod_{s \in S} \left(\sum_{j \in \mu'(s)} w_{js} \right)^{r_s}.$$

As all schools are occupied at μ^*, we can divide both sides by all factors not pertaining to schools s and s' and get

$$\left(\sum_{j \in \mu^*(s)} w_{js} \right)^{r_s} \left(\sum_{j \in \mu^*(s')} w_{js'} \right)^{r_{s'}} \geq \left(\sum_{j \in \mu'(s)} w_{js} \right)^{r_s} \left(\sum_{j \in \mu'(s')} w_{js'} \right)^{r_{s'}}.$$

As $i \in \mu'(s')$ the term $\left(\sum_{j \in \mu'(s')} w_{js'} \right)^{r_{s'}}$ is nonzero as well. Dividing both parts by it and by $\left(\sum_{j \in \mu^*(s)} w_{js} \right)^{r_s}$, we get

$$\left(\frac{\sum_{j \in \mu^*(s')} w_{js'}}{\sum_{j \in \mu'(s')} w_{js'}} \right)^{r_{s'}} \geq \left(\frac{\sum_{j \in \mu'(s)} w_{js}}{\sum_{j \in \mu^*(s)} w_{js}} \right)^{r_s}$$

$$\left(1 - \frac{w_{is'}}{\sum_{j \in \mu'(s')} w_{js'}} \right)^{r_{s'}} \geq \left(1 - \frac{w_{is}}{\sum_{j \in \mu^*(s)} w_{js}} \right)^{r_s}. \tag{23}$$

For each school s, define the function $g_s(x) := (1-x)^{r_s}$. Let β_{is} be the share of the resource that student i gets at school s at μ^* and $\beta_{is'}$ be the share of the resource that student i gets at school s' at μ'. With this notation (23) can be rewritten as

$$g_{s'}(\beta_{is'}) \geq g_s(\beta_{is}).$$

Now we perform the key step: we replace g_s and $g_{s'}$ with their first-order Taylor expansions around $x = 0$ using the Lagrange form of the remainders. We obtain

$$1 - r_{s'}\beta_{is'} + g''(c_{is'})\beta_{is'}^2/2 \geq 1 - r_s\beta_{is} + g''(c_{is})\beta_{is}^2/2,$$

where $c_{is} \in [0, \beta_{is}]$, $c_{is'} \in [0, \beta_{is'}]$. Rearranging, we get

$$r_s\beta_{is} - r_{s'}\beta_{is'} \geq \frac{1}{2}\left(g_s''(c_{is})\beta_{is}^2 - g_{s'}''(c_{is'})\beta_{is'}^2\right).$$

Now notice that $r_s\beta_{is} = u_i(s_i, s_{-i})$, $r_s\beta_{is'} = u_i(s', s_{-i})$. Thus, we know that

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \geq \frac{1}{2}\left(g_s''(c_{is})\beta_{is}^2 - g_{s'}''(c_{is'})\beta_{is'}^2\right). \tag{24}$$

At the same time, to establish that μ_n^* is an approximate Nash equilibrium we need to show that

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) \geq -\varepsilon u_i(s_i, s_{-i}), \tag{25}$$

which is the slightly rewritten definition (22). Thus, if we prove that for all sufficiently large n and all i, s_i, s_i'

$$\frac{1}{2}\left(g_s''(c_{is})\beta_{is}^2 - g_{s'}''(c_{is'})\beta_{is'}^2\right) \geq -\varepsilon u_i(s_i, s_{-i}), \tag{26}$$

this, together with (24), would imply that the desired result (25). So it remains to show (26), which may be rewritten as

$$\frac{1}{2}\left(g_s''(c_{is})\beta_{is}^2 - g_{s'}''(c_{is'})\beta_{is'}^2\right) \geq -\varepsilon r_s\beta_{is}.$$

Dividing both sides by β_{is}, which is nonzero as $w_{is} > 0$, we get

$$\frac{1}{2}\left(g_s''(c_{is})\beta_{is} - g_{s'}''(c_{is'})\beta_{is'}\frac{\beta_{is'}}{\beta_{is}}\right) \geq -\varepsilon r_s. \tag{27}$$

Now our goal is to show that, given $\varepsilon > 0$, there exists a number $N_{s,s'}(\varepsilon)$, depending only on pair of schools s, s' and ε but not on i, such that (27) holds for all $n > N_{s,s'}(\varepsilon)$. To this end, we will show that the LHS of (27) converges to 0 *uniformly over* i as $n \to \infty$. Without the uniform convergence we would get a bound $N_{i,s,s'}(\varepsilon)$ depending on i as well; taking then the maximum of $N_{i,s,s'}(\varepsilon)$ over all i, s, s' would be problematic since the number of students at school s may grow arbitrarily large.

Our strategy is to show that in (27): (i) β_{is} converges to 0 uniformly over i as $n \to \infty$; (ii) $\beta_{is'}$ converges to 0 uniformly over i as $n \to \infty$; (iii) for any two schools s, s' and student i, the ratio $\frac{\beta_{is'}}{\beta_{is}}$ stays uniformly bounded over i as $n \to \infty$. Then the result will follow, as $c_{is} \in [0, \beta_{is}]$ and $g''(0)$ is finite. The ratio $\frac{\beta_{is'}}{\beta_{is}}$ may in principle be unbounded if the numbers of students at s and s' at μ_n^* have different rates of growth. Fortunately, it may be proved that due to the fact that μ_n^* is not an arbitrary strategy profile but the one maximizing

$\tilde{P}(\mu)$, this may not happen — the number of students at *every* school at μ_n^* must grow linearly in n for large n, which is equivalent to saying that the share of students at *every* school stays positive in the limit. We show this in the following lemma.

Lemma 1. *Denote $\alpha_s^*(n) := \frac{n_s^*}{n}$, where n_s^* is the number of students at school s in the matching μ_n^* maximizing $\tilde{P}$ (with the sequence of matrices W^n fixed). Then, for every school s*

$$\liminf_{n\to\infty} \alpha_s^*(n) > 0. \tag{28}$$

Proof. See Appendix. □

Lemma 1 implies that for every school s there exists numbers $d_s > 0$ and M_s such that for all $n > M_s$ $n_s^*/n > d_s$. But then for all i and s we have

$$\beta_{is} \equiv \frac{w_{is}}{\sum_{j\in\mu^*(s)} w_{js}} \le \frac{\overline{w}}{\underline{w} n_s^*} < \frac{\overline{w}}{\underline{w} d_s n},$$

for all $n > M_s$, so β_{is} converges to 0 uniformly over i as $n \to \infty$. Also,

$$\beta_{is'} \equiv \frac{w_{is}}{\sum_{j\in\mu'(s)} w_{js}} \le \frac{\overline{w}}{\underline{w}(n_{s'}^* + 1)} < \frac{\overline{w}}{\underline{w}(d_{s'} n + 1)},$$

for all $n > M_{s'}$, so $\beta_{is'}$ also converges to 0 uniformly over i as $n \to \infty$. (We have $n_{s'}^* + 1$ rather than $n_{s'}^*$ in the denominator because $\beta_{is'}$ is the share of school's s' resource the student i gets upon deviation from s at μ_n^* to s'.)

Finally,

$$\frac{\beta_{is'}}{\beta_{is}} \le \frac{\frac{\overline{w}}{\underline{w}(n_{s'}^*+1)}}{\frac{\underline{w}}{\overline{w} n_s^*}} \le \left(\frac{\overline{w}}{\underline{w}}\right)^2 \frac{n_s^*}{n_{s'}^* + 1} \le \left(\frac{\overline{w}}{\underline{w}}\right)^2 \frac{n}{d_{s'} n + 1} \le \left(\frac{\overline{w}}{\underline{w}}\right)^2 \frac{1}{d_{s'}}$$

for $n > M_{s'}$, where we have used $n_s^* \ge n$ and $n_{s'}^* > d_{s'} n$ for the third inequality. Thus, $\frac{\beta_{is'}}{\beta_{is}}$ is bounded from above uniformly over i as $n \to \infty$.

As the constants c_{is} and $c_{is'}$ in (27) satisfy $0 \le c_{is} \le \beta_{is}$, $0 \le c_{is'} \le \beta_{is'}$, it follows that c_{is} and $c_{is'}$ also converge to 0 uniformly over i as $n \to \infty$.

Combining the facts that $\beta_{is}, \beta_{is'}, c_s, c_{s'}$ converge to 0 uniformly in i as $n \to \infty$, the fact that $\frac{\beta_{is'}}{\beta_{is}}$ is bounded from above uniformly over i as $n \to \infty$, and the fact that $g_s''(0)$ is finite, we obtain that the LHS of (27) converges to 0 uniformly in i as $n \to \infty$. This means that for every $\varepsilon > 0$ there exists a number $N_{s,s'}(\varepsilon)$, depending only on pair of schools s, s' and ε such that (27) holds for all $n > N_{s,s'}(\varepsilon)$.

But then for all $n > N(\varepsilon) := \max\{\max_{s,s'}\{N_{s,s'}(\varepsilon)\}, |S|\}$ (recall that we consider only $n \ge |S|$ from the outset of this proof) the inequalities (27) hold for every pair of schools s, s', and thus μ_n^* is an approximate Nash equilibrium with relative error at most ε. □

Two comments about Theorem 4 are in order.

- Note that if a profile of strategies is an approximate Nash equilibrium with relative error at most $\varepsilon > 0$, it will remain such when all the resources are increased by any factor. Thus, by Theorem 4 an approximate NE still exists when the resources at all schools are increased (at any rate) together with the growth of the total student population, which may be more realistic asymptotics to consider.
- This result differs from the one in, e.g., Leshno (2022) in that it does not require the weights to be generated by an iid sampling process from a continual distribution. This is what the "approximate potential" method buys us. And even if the weights were generated by a sampling process, Theorem 4 shows the existence of an approximate Nash equilibrium with probability 1 rather than $1 - \varepsilon$ as in Leshno (2022).

4. Conclusion

The existence of a pure-strategy Nash equilibrium in a general setting — arbitrary weights, arbitrary resources, and more than two schools– remains elusive. But we conjecture that it always exists; we also conjecture that the game has the finite improvement property in the general case. The theoretical proof however remains remarkably infeasible despite many attempts and approaches we have tried, including those that worked successfully for the special cases presented in the paper. We encourage the reader to try to prove the existence in a general case, but we do so with restraint. We also encourage the reader to think about potential synergies between the literatures on congestion games and matching with externalities that may further develop the link we discussed in the literature review.

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Appendix

Proof of Lemma 1. Suppose, by way of contradiction, that for some school k $\liminf_{n\to\infty} \alpha_k^*(n) = 0$. As for all i and s $w_{is} < \overline{w}$, the value of the approximate potential (18) at optimum, $P(\mu_n^*)$, satisfies

$$P(\mu_n^*) \leq \prod_{s\in S} (\overline{w} n_s^*)^{r_s} = n^{\sum_s r_s} \prod_{s\in S} (\overline{w} \alpha_s^*(n))^{r_s}.$$

Because $\liminf_{n\to\infty} \alpha_k^*(n) = 0$ for school k and all $\alpha_s^*(n)$ are bounded from above by one we have

$$\liminf_{n\to\infty} \frac{P(\mu_n^*)}{n^{\sum_s r_s}} = 0. \tag{29}$$

But now consider any sequence of strategy profiles $\hat{\mu}_n$ such that the share of students going to each school is bounded away from zero for all sufficiently large n (for example, send approximately equal numbers of students to each school). Denote the corresponding numbers and shares of students $\hat{n}_s$ and $\hat{\alpha}_s(n)$. By construction, for all $s \in S$, $\liminf_{n\to\infty} \hat{\alpha}_s(n) > 0$. We have

$$\tilde{P}(\mu_n^*) \geq \tilde{P}(\hat{\mu}_n) \geq \prod_{s\in S} (\underline{w} \hat{n}_s)^{r_s} = n^{\sum_s r_s} \prod_{s\in S} (\underline{w} \hat{\alpha}_s(n))^{r_s},$$

where the first inequality is by the optimality of μ_n^*, the second inequality by $w_{is} \geq \underline{w}$ and the definition of $\tilde{P}$ (18), and the equality is by the definition of $\hat{\alpha}_s(n)$. Thus,

$$\liminf_{n\to\infty} \frac{P(\mu_n^*)}{n^{\sum_s r_s}} = \liminf_{n\to\infty} \prod_{s\in S} (\underline{w} \hat{\alpha}_s(n))^{r_s} > 0, \tag{30}$$

which contradicts (29).