Stable coalition structure for transversal problems

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Many actors in the economy, management, sociology, and other spheres have qualifications. Agents develop in a certain direction to become specialists. To be able to implement some projects or assignments, however, representatives of different qualifications or groups have to form coalitions by collaborating. Each coalition member performs the job he/she specializes in. Then, this system of distinct representatives gets a certain utility. The system of distinct representatives, or the transversal, is a coalition of agents from different groups.

Suppose we have two groups: theorists and practitioners. Working together, a theorist and a practitioner can come up with an application for a theory. A coalition of a theorist and a practitioner is a transversal. A buyer and seller, and a sender and receiver are also examples of transversals. An agent gains utility only if he/she belongs to a transversal.

Formally, let N be a non-empty finite set of agents and $\pi = \{B_1, B_2, ..., B_l\}$ be a partition of the agents into disjoint non-empty groups. Then, the transversal of partition π is a coalition $K = \{t_1, t_2, ..., t_l\}$, where $t_j \in B_j \ \forall j \in \{1, 2, ..., l\}$. Consider two situations where transversals are important:

1. The formation of workgroups. Let N be the set of workers and l be the number of types of jobs to be fulfilled to implement one project. The set of workers is to be portioned into l groups so that one type of job is associated with each group. Then, any transversal of the resulting partition is a minimal coalition of workers able to implement the project.

2. The appointment of chairpersons. An institution needs to form l commissions, but the same person cannot be member of two commissions at a time. Chairpersons are appointed from among commission members. In this case, any coalition of chairpersons is a transversal of the partition.

Workgroup formation can result in any partition into l non-empty coalitions, i.e. $|\pi| = l$. In the following, workers form transversals and derive some utility thereby. Each worker is interested in getting as much profit from the eventual partition as possible. Before transversals are formed, workers can move between coalitions of the partition, meaning that a worker can shift from one type of job to another. In this case, the question arises of whether a stable coalition structure exists. A similar issue appears in the problem of chairpersons, because each agent is interested in chairing the respective commission.

The problem of forming working groups and the problem of appointing chairpersons differ from each other in the context, but both boil down to solving transversal problems.

The utility of a coalition in this paper is determined using cooperative game theory. The transversal value is introduced for cooperative games with coalition structure. The transversal value of player $i, i \in N$ for a given partition π is the sum of the values of partition π, the transversals of which player i is a member. Coalition values are determined by the characteristic function $v, v : 2^N \to \mathbb{R}$, which is independent of partition π. A feature of the transversal value of player i in partition π is that it does not depend on other players in the coalition that player i is a member of. This is a significant distinction from many other game-theoretic models.

The article introduces the workgroup formation game and the game of chairpersons. They are novel games of coalition partition, where players' payoffs are expressed through transversal values.

Players in the workgroup formation game are the firm owner and workers. Workers form a coalition partition which excludes the firm owner. The characteristic function v indicates the number of projects that can be implemented by the coalition $K, K \subseteq N$. When the partition is formed, workers form transversals to implement the projects. The firm owner gets a certain amount of money from each project implemented. The owner then pays the workers for the completed projects in which they participated. A Pareto-Nash-stable coalition structure is shown to exist in the workgroup formation game. This means that workers can be split into a fixed number of groups so that none of them will want to individually move to other groups and the owner's payoff will be maximized.

The characteristic function of the game of chairpersons represents *a priori* probabilities that a certain coalition is a coalition of chairpersons. Each player is interested in a partition in which his/her odds of becoming a chairperson are high. A player's payoff in the game of chairpersons is the probability of becoming the leader in his/her coalition, given that partition π is formed. We are thus dealing with a random variable that takes value 1 or 0. In this case, the expected value of such a random variable is the probability of being a leader for partition π. A Nash-stable coalition structure is shown to exist in the game of chairpersons.

Players' payoff functions in the workgroup formation game and the game of chairpersons are expressed in a special manner through the transversal value and its potential function. Hence, properties of the transversal value are applied to the players' payoff functions in these games.

The study produced the following results:

- We prove that any cooperative game has a coalition structure that is simultaneously Nash stable and permutation stable, and has a total payoff maximization (TPM) for the transversal value. If values of the characteristic function are non-negative, the coalition structure with these three types of stability is also externally stable.
- Punctual stability is introduced for the coalition structures that maximize the potential function. This type of stability is studied for the transversal value.
- A Pareto-Nash-stable coalition structure is proved to exist in the workgroup formation game.
- The game of chairpersons is shown to be an ordinal potential game. Punctual stability is studied for the game of chairpersons.

The existence of a coalition structure that is simultaneously Nash stable and permutation stable is guaranteed by the fact that the potential functions for the different types of stability coincide. A similar result can be found in (Gusev, 2021)[1]. (Gusev, 2021) examined games where a player's payoff depended on his/her coalition, whereas the transversal value did not have this property. Game-theoretic models from (Gusev, 2021) and in the present paper do not follow from one another. Furthermore, there exists a coalition structure with two types of stability which is TPM for the transversal value. Where values of the characteristic function are non-negative, there is one more type of stability. No such result is found in (Gusev, 2021).

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[1]Gusev, V. V. (2021). Nash-stable coalition partition and potential functions in games with coalition structure. *European Journal of Operational Research*, 295(3), 1180-1188.